View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Open Access Repository

Physics Letters B 729 (2014) 136–142

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Shock wave evolution and discontinuity propagation for relativistic superfluid hydrodynamics with spontaneous symmetry breaking ☆

Sun Zhang [a,b,c,*]

[a] Purple Mountain Observatory, Chinese Academy of Sciences, Nanjing 210008, China
[b] Key Laboratory of Dark Matter and Space Astronomy, Chinese Academy of Sciences, Nanjing 210008, China
[c] Joint Center for Particle, Nuclear Physics and Cosmology (J-CPNPC), PMO-NJU, Nanjing 210008, China

ARTICLE INFO

Article history:
Received 24 October 2013
Received in revised form 17 November 2013
Accepted 7 January 2014
Available online 10 January 2014
Editor: M. Cvetič

Keywords:
Relativistic superfluid
Hydrodynamics
Shock wave
Sound speed

ABSTRACT

In this Letter, we have studied the shock wave and discontinuity propagation for relativistic superfluid with spontaneous $U(1)$ symmetry breaking in the framework of hydrodynamics. General features of shock waves are provided, the propagation of discontinuity and the sound modes of shock waves are also presented. The first sound and the second sound are identified as the propagation of discontinuity, and the results are in agreement with earlier theoretical studies. Moreover, a differential equation, called the growth equation, is obtained to describe the decay and growth of the discontinuity propagating along its normal trajectory. The solution is in an integral form and special cases of diverging waves are also discussed.

© 2014 The Author. Published by Elsevier B.V. All rights reserved.

1. Introduction

In 1941, Landau proposed his famous two-fluid (or two-constituent) theory for superfluid helium [1,2]. The basic idea of the theory is that liquid helium behaves as if it were a mixture of two distinguished liquids when temperature is below the critical point in a local equilibrium state. That is, one of them is a normal viscous fluid (or excitation microscopically) with entropy and thermal conductivity, while the other is a non-viscous fluid, called superfluid (or condensate microscopically) which can move dissipationlessly along a solid surface with zero viscosity. No momentum is assumed to transfer from one to the other, and no viscous communication takes place between these two components. In fact, the two-fluid picture was presented by Tisza firstly [3] inspired by Bose condensate in momentum space, and studied extensively and intensively after Landau's work [4–7]. Phenomenologically and semi-microscopically, the two-fluid theory is now accepted as a fundamental model for the description of superfluid hydrodynamics. In the non-relativistic frame, such a theory was studied and summarized by Khalatnikov [7].

The first approach to the theory of relativistic superfluid was provided by Israel [8] and Dixon [9] based on the idea of two-fluid picture, treating both constituents as perfect fluids. The medium is usually called relativistic in two senses: if it has a relativistic equation of state or when it flows at a relativistic velocity [10]. Both conditions can be satisfied in the massive neutron star [11–15] whose core could be considered as the construction of superfluid nuclear matter [16], superfluid nucleon–hyperon mixture [13] and/or paring quark matter [17].

It is also well known that constituents of superfluid are not perfect fluids strictly and both of them cannot be regarded as completely independent fluids [18,19]. Especially in the relativistic regime the system can be a strongly self-interacting one [19]. That is, the coupling effects would drive the medium deviated far from ideality [10]. At finite temperature, the communication (or entrainment) between the superfluid component (or condensate) and the normal entropy component (or thermal excitations) can play a crucial role in some circumstances [14]. Then it is meaningful to take into account the deviation from the perfect fluids in order to obtain more effective models. Khalatnikov and Lebedev [18,20] and Carter and Khalatnikov [21] suggested two equivalent approaches, the so-called potential and convective variational models respectively [22], to include the interaction between the superfluid and normal fluid.

As pointed out in Ref. [23], a superfluid obeys hydrodynamics microscopically while displays quantum effects macroscopically. From the viewpoint of field theory, a condensate phase is

☆ This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/3.0/). Funded by SCOAP[3].

* Correspondence to: Purple Mountain Observatory, Chinese Academy of Sciences, Nanjing 210008, China.
E-mail address: szhang@pmo.ac.cn.

0370-2693/$ – see front matter © 2014 The Author. Published by Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.physletb.2014.01.014

associated with spontaneous breaking of $U(1)$ global symmetry. Son [24] presented another approach to the relativistic superfluid with $U(1)$ symmetry breaking based on the Poisson bracket technique [25,26]. Such a description is also arisen from the two-constituent theory, and is effective to include the long-range Goldstone modes and more clearer to show the relation between superfluid and symmetry breaking. After constructing some basic relations of the fundamental quantities by Poisson brackets, including fluid (thermo-) ones and field ones, the hydrodynamic equations are obtained naturally, which provide the explicit meaning associated with the symmetry breaking. It can be used directly to the relativistic superfluid to study the dissipative effects [27,28] and anomalies [29]. Furthermore, recently this superfluid hydrodynamic framework with spontaneous symmetry breaking has been applied to holographic models by Herzog, Kovtun, Son, Yarom et al. [30–34]. Using AdS/CFT correspondence, they try to construct the condensate phase or superfluid in AdS geometry asymptotically and study the hydrodynamics, including sound modes and possible phase transitions in strongly interacting relativistic superfluid. Alford et al. [22,35] attempt to relate various "macroscopic" two-constituent models to an underlying "microscopic" field theory with a global $U(1)$ symmetry spontaneous breaking. In their framework, using ϕ^4-model of a complex scalar field, the entrainment coefficient and sound speeds are calculated, and the relationship between different formulations of various two-constituent models is also discussed. The two-constituent or multi-constituent [36] theory is appealing and still in development [37].

On the other hand, the shock waves or discontinuity phenomena are widely existent in hydrodynamic systems [38], including superfluid [7]. Significantly, the relativistic shock processes can play important roles in the heavy-ion collisions at high energies and in the early universe [39,40]. The production and evolution of the hot dense matter, quark–gluon plasma are interested theoretically and experimentally. While the experiments of RHIC at Brookhaven and LHC of CERN at Geneva can provide us important events and data, the hydrodynamic framework is shown to be efficient to describe not only the bulk fluid evolution but also the propagation of discontinuity and sonic (or supersonic) waves.

To the best of the author's knowledge, the first work on the shock wave of relativistic superfluid was given by Carter in Ref. [41]. Inspired by Hadamard's method, the weak discontinuities are introduced to the space–time derivatives of physical quantities, instead of these quantities themselves, when crossing some characteristic surface. Two propagation modes of the discontinuities are shown as a "heat" (or second sound) mode and an "ordinary" (or first sound) mode respectively. After Carter's work, some other researches appeared in the same theme [10,42]. Additionally, Vlasov's works [10] are not based on the Hadamard's method to some extent, and the shock waves in superconductive cosmic strings are also discussed [43].

In this Letter, we shall study the shock wave and discontinuity propagation for the relativistic superfluid with spontaneous $U(1)$ symmetry breaking. The following sections are organized as follows. The general features of shock waves are presented in Section 2. While the propagation of discontinuity and the sound modes of shock waves are discussed in Section 3, the first sound and the second sound are identified in agreement with the earlier work by Herzog et al. [30]. Furthermore, in Section 4, the growth equation, governing the decay and growth of the discontinuity propagation along its normal trajectory, is also obtained, and special cases of diverging waves are also examined for the solution of the growth equation. Finally, we would like to give a summary in Section 5.

2. General features of shock waves for relativistic superfluid

In the framework of relativistic hydrodynamics, shock phenomena have been widely studied especially for its evolution and phase transition in heavy-ion collisions [39,40,44–47]. The hydrodynamic picture on these processes is physically clear and simple, only the conservation laws for energy–momentum and baryon number are required in the beginning. Under the initial conditions the behaviors of the discontinuities crossing some singular surface are governed by the conservation laws and can give a reasonable description for the shock phenomena. Moreover, such a framework can be generalized naturally to more complicated systems, such as multi-component fluid [48], some quark pairing superfluid [49] and so on. These systems can also be studied along the hydrodynamic approach. One of the central issues is on the $U(1)$ symmetry breaking. The related current of $U(1)$ will introduced to the charge conservation equation and the contribution of the condensate also should be included to the energy–momentum tensor. Furthermore, an extra equation would appear to relate the condensate phase and some hydrodynamic driving potential, such as the chemical potential μ in superfluidity or the scalar potential V in superconductivity. Such a conjugated equation is the Josephson-type equation [50].

To study the general features of shock waves for relativistic superfluids, we briefly rewrite the main results of the superfluid hydrodynamics by Son [24,30]. We take the metric tensor $\eta^{\mu\nu}$ as $\mathrm{diag}(-1,+1,+1,+1)$, then the four-velocity u^μ is normalized as $\eta_{\mu\nu}u^\mu u^\nu=-1$. The equations for energy–momentum conservation and $U(1)$ charge conservation are

$$\partial_\mu T^{\mu\nu}=0, \tag{1}$$

and

$$\partial_\mu j^\mu=0, \tag{2}$$

where $T^{\mu\nu}$ and j^μ are clearly expressed by the addition of two components, the normal one and the symmetry breaking one (or superfluid one),

$$T^{\mu\nu}=(\epsilon+P)u^\mu u^\nu+P\eta^{\mu\nu}+f^2\partial^\mu\varphi\partial^\nu\varphi, \tag{3}$$

and

$$j^\mu=nu^\mu+f^2\partial^\mu\varphi, \tag{4}$$

and the Josephson equation is also obtained,

$$u^\mu\partial_\mu\varphi+\mu=0. \tag{5}$$

The energy density ϵ is defined by $\epsilon=TS+n\mu-P$, a Legendre transformation of P, the velocities of the normal fluid component and the superfluid component are identified respectively as

$$u^\mu=(1,\mathbf{v})/\sqrt{1-v^2},\qquad u_s^\mu=\partial^\mu\varphi/\mu\equiv\xi^\mu/\mu. \tag{6}$$

The normalized u_μ is the time-like four velocity of the normal component and associated with the entropy flow, while $\mathbf{v}$ is the three dimensional velocity $v_k=\partial x_k/\partial t$. The equation of state, describing the relation of pressure P, temperature T, chemical potential μ and phase φ could be written in the differential form as

$$dP=s\,dT+n\,d\mu-f^2\xi^\mu\,d\xi_\mu. \tag{7}$$

The conservation of entropy $\partial_\mu(su^\mu)=0$, with no dissipative transports in the system, can be derived from above and is not an independent equation.

When the flow passes through some surface Σ, there exists discontinuities in certain physical quantities of the system. The particle charge and the energy–momentum flux should be continuous from one side to the other side of the surface [2,39], and the propagation of the matter front exhibits the characteristic behavior of shock wave. We call the side 2 (side 1) that behind (before) the surface, then the difference of the values of some quantity Z on the two sides of the surface is denoted by

$$[Z]=Z_2-Z_1. \tag{8}$$

These notations are following Landau and Lifshitz [2] and Csernai [39], but somewhat different from those of Vlasov [10]. Meanwhile, we define the surface Σ a time-like (space-like) one when its unit normal vector N^μ is space-like $N^\mu N_\mu=1$ (time-like $N^\mu N_\mu=-1$). Such a definition is in agreement with that of Thomas [51] and Bugaev et al. [45], but different from that of Csernai [39]. We concentrate ourselves on the time-like surface Σ only, where the shock front velocity is smaller than light speed. Then the conservation laws can be expressed as

$$\left[T^{\mu\nu}N_\mu\right]=0, \qquad \left[j^\mu N_\mu\right]=0, \tag{9}$$

when the flow crosses the surface of the discontinuity. If we set the shock front at rest, $N_\mu=(0,1)$, the above shock equations should be

$$T_1^{01}=T_2^{01}, \qquad T_1^{11}=T_2^{11}, \qquad j_1^1=j_2^1. \tag{10}$$

Then we can obtain fluid velocities on both sides of the discontinuity v_1^2 and v_2^2

$$v_1^2=\frac{(P_2-P_1)(P_1+\epsilon_2+\mu_2^2f^2)}{(\epsilon_2-\epsilon_1+\mu_2^2f^2-\mu_1^2f^2)(P_2+\epsilon_1+\mu_1^2f^2)}, \tag{11}$$

$$v_2^2=\frac{(P_1-P_2)(P_2+\epsilon_1+\mu_1^2f^2)}{(\epsilon_1-\epsilon_2+\mu_1^2f^2-\mu_2^2f^2)(P_1+\epsilon_2+\mu_2^2f^2)}. \tag{12}$$

As stated above, v_1 (v_2) is for the three dimensional velocity or spatial part of u_μ on side 1 (2). Eqs. (11) and (12) are directly from the conservation laws and could be used further. Defining $k_i=n_i+\mu_if^2$, $i=1$ (2) for side 1 (2), together with the equation of flux conservation, we obtain

$$k_2^2X_2^2-k_1^2X_1^2-(P_2-P_1)(X_1+X_2)=0, \tag{13}$$

where

$$X_i\equiv\frac{\epsilon_i+P_i+\mu_i^2f^2}{(n_i+\mu_if^2)^2}, \quad i=1\,(2)\text{ for side }1\,(2) \tag{14}$$

is the generalized specific volume [39]. It is clear that Eq. (13) is the Taub adiabat [52] in its generalized form of relativistic superfluid, which gives the relation of discontinuous quantities and describes the hydrodynamic evolution of the system during crossing the shock surface. If we define $x=X_2/X_1$,

$$\omega_2x-\omega_1-(P_2-P_1)(1+x)=0, \tag{15}$$

where $\omega_i=\epsilon_i+P_i+\mu_i^2f^2$, $i=1$ (2) for side 1 (2). Clearly, this is an equation related the thermodynamic variables P, x and ω. If the equation of state (EOS) is provided, we can obtain a curve in the (P, x) plane to describe the evolution of shock waves from initial state "1" to the final state "2". The shock adiabat is determined not only by the EOS of side 1 but also by that of side 2. If both sides are with the same EOS, there is the normal shock. If the EOS of one side is different from that of the other because of some thermodynamic and/or mechanic interactions, a phase transition could be taken place while crossing the shock surface Σ. Based on these, we now discuss concisely three cases for the relativistic fluid and/or superfluid with certain ideal gas EOS and/or conformal EOS.

If we take the fluid to be an ideal relativistic one, its EOS is $P=\frac{1}{3}\epsilon$ [40], and the shock adiabat equation would be

$$4P_2x-4P_1-(P_2-P_1)(1+x)=0, \tag{16}$$

that is,

$$P_2=P_1\frac{3-x}{3x-1}, \quad \text{or} \quad P_1=P_2\frac{3x-1}{3-x}, \tag{17}$$

which is discussed in detail in Ref. [40].

If we take the fluid on side 1 to be an ideal-like relativistic one while the fluid on side 2 to be a superfluid, with the scaling-invariant or conformal EOS $P=T^4g(\mu/T)$ [30,31,33,34], we can describe a transition from one to the other based on the shock adiabat as

$$\left(4P_2+\mu_2^2f^2\right)x-4P_1-(P_2-P_1)(1+x)=0, \tag{18}$$

that is,

$$P_2=P_1\frac{3-x}{3x-1}-\frac{\mu_2^2f^2x}{3x-1}. \tag{19}$$

Similarly, if we take the fluid to be a relativistic superfluid, the shock adiabat equation would be

$$\left(4P_2+\mu_2^2f^2\right)x-\left(4P_1+\mu_1^2f^2\right)-(P_2-P_1)(1+x)=0, \tag{20}$$

that is,

$$P_2=P_1\frac{3-x}{3x-1}-\frac{\mu_2^2f^2x-\mu_1^2f^2}{3x-1}. \tag{21}$$

Eq. (20) can also be used to discuss the transition between two different superfluidity phases, such as the phase A and phase B of superfluid ^{3}He on a moving phase boundary [53]. Evidently, the chemical potential plays a crucial role in these transitions.

We can also introduce the Poisson adiabat for relativistic superfluid to describe the entropy flux,

$$R^2=\frac{s_2^2}{s_1^2}x\frac{\omega_1}{\omega_2}, \tag{22}$$

where s_i is the entropy in the local rest frame, $i=1$ (2) for side 1 (2). In the so-called large-entropy limit $Ts\gg\mu n$ [30,31], the Poisson adiabat can be written as

$$\frac{s_2}{T_2}=\frac{R^2}{x}\frac{s_1}{T_1}, \tag{23}$$

for the shock front of relativistic superfluid.

Consequently, in the relativistic superfluid systems, various shock transitions can be contained in one unified hydrodynamic framework based on adiabats. These transitions are associated with diverse phase transitions via miscellaneous equations of states. We can reach a qualitative description basically, and also some quantitative results by means of adiabats. Further studies can make these points more clearer.

3. Propagation of discontinuity, the first sound and the second sound

The comprehensive approach to the study of surface of discontinuity in continuum media was developed by Thomas [54,51,55] (see also Ref. [56]). Thanks to these works, the singular surface theory has been promoted to an applicable and effective level in its present form. Based on this, many studies were performed to various classes of materials to describe the properties of the discontinuities and shock waves. After the works on the propagation of weak discontinuities in ideal gases [57–60] assuming the media ahead of the wave in the uniform state, much interest arose in the study of anisotropic wave propagation in non-uniform media [61–65]. It is known that the propagation of the discontinuity would be anisotropic if the medium enters into a non-uniform region [2]. The basic research on this topic was proposed by Elcrat [66] for an unsteady flow in the non-equilibrium dynamic system. On the other hand, a mathematical theory of geometric optics, the ray theory, was also be applied to the singular surface theory to study the propagation of discontinuities in non-linear anisotropic media [67–69]. Especially, the singular surface theory has many important applications in relativistic systems, such as the initially stressed relativistic elastic solid [70], the relativistic viscous fluid [71], and the hot relativistic plasma [72]. A meaningful study of the shock wave propagation in relativistic magnetohydrodynamics was given by Lichnerowicz in terms of the distribution theory of generalized functions [73].

In relativistic space–time, we consider, adopting the notations similar to those of Ref. [68,72], a propagating, time-like, singular surface Σ represented by either of the equations

$$f\left(x^\alpha\right)=0, \quad x^\alpha=x^\alpha\left(z^A\right), \tag{24}$$

where z^A (capital Latin indices assume the values $0, 1, 2$) are the coordinates on Σ. Following Thomas [54,51,57], a moving surface is defined to be singular of order 1 relative to the physical quantities, such as pressure, temperature, density and so on, if these quantities are continuous across the surface but the first derivatives of them with respect to the coordinates x^α are discontinuous. That is, using the jump bracket of Eq. (8), for the singular surface of order 1,

$$[Z]=0, \qquad [Z_{,\alpha}]\neq 0. \tag{25}$$

If we take a_{AB} and b_{AB} to be the components of the first and second fundamental covariant tensors of Σ respectively, the following equations are obtained [54,51,56,74]

$$\begin{aligned}
&N^\alpha_{,\alpha}=-a^{AB}b_{AB}, \qquad N^\alpha_{,A}=-b_{AD}a^{BD}x^\alpha_{,B}, \qquad N_\alpha x^\alpha_{,A}=0,\\
&x^\alpha_{,AB}=b_{AB}N^\alpha, \qquad a^{AB}x^\alpha_{,A}x^\beta_{,B}=g^{\alpha\beta}-N^\alpha N^\beta,\\
&a_{AB}=g_{\alpha\beta}x^\alpha_{,A}x^\beta_{,B}=a_{BA},
\end{aligned} \tag{26}$$

with notations

$$b^A_B\equiv a^{AC}b_{CB}, \qquad x_{\alpha,A}\equiv g_{\alpha\beta}x^\beta_{,A}, \tag{27}$$

where N_α is the unit normal vector of the singular surface Σ, defined by a normalized form

$$N_\alpha=\frac{f_{,\alpha}}{(\eta^{\mu\nu}f_{,\mu}f_{,\nu})^{1/2}}, \qquad N_\alpha N^\alpha=1. \tag{28}$$

As stated above, the case here we considered is a time-like surface Σ with the space-like normal vector N_α. Moreover, vectors $x^\alpha_{,A}$ are tangent to the surface, so $N_\alpha x^\alpha_{,A}=0$ in Eq. (26).

The compatibility conditions which must be satisfied across Σ by the first and second partial derivatives

$$[Z_{,\alpha}]=\Gamma N_\alpha, \tag{29}$$

$$\begin{aligned}
[Z_{,\alpha\beta}]=&\,\overline{\Gamma}N_\alpha N_\beta+a^{AB}\Gamma_{,A}(N_\alpha x_{\beta,B}+N_\beta x_{\alpha,B})\\
&-\Gamma b^{AB}x_{\alpha,A}x_{\beta,B},
\end{aligned} \tag{30}$$

where Γ and $\overline{\Gamma}$ are notations of the discontinuities in the normal derivatives, that is, $[Z_{,\alpha}]N^\alpha=\Gamma$ and $[Z_{,\alpha\beta}]N^\alpha N^\beta=\overline{\Gamma}$.

If we look on the temperature T, the chemical potential μ, the normal fluid velocity u^μ and the superfluid velocity ξ^μ as four independent variables, as in Ref. [30], the discontinuities in the normal derivatives in terms of the jump bracket for these variables are

$$\begin{aligned}
&\left[\frac{\partial T}{\partial x^\alpha}\right]N^\alpha=\chi, \qquad \left[\frac{\partial \mu}{\partial x^\alpha}\right]N^\alpha=\psi,\\
&\left[\frac{\partial u^\alpha}{\partial x^\beta}\right]N^\beta=\omega^\alpha, \qquad \left[\frac{\partial \xi^\alpha}{\partial x^\beta}\right]N^\beta=\upsilon^\alpha.
\end{aligned} \tag{31}$$

To proceed further, we make a decomposition on the conservation equation of energy–momentum tensor $\partial_\mu T^{\mu\nu}=0$. The projection tensor $\Delta^{\mu\nu}$ is defined in the following way

$$\Delta^{\mu\nu}=g^{\mu\nu}+u^\mu u^\nu, \tag{32}$$

then $\Delta^{\mu\nu}$ is perpendicular to u^μ in the sense $u_\mu\Delta^{\mu\nu}=0$. It is clear that $\Delta^{\mu\nu}$ (u^μ) picks up the space-like (time-like) component when acting on some Lorentz vector/tensor in the local rest frame, see e.g. Refs. [75,76]. In the case here, we use u^μ and $\Delta^{\mu\nu}$ to decompose Eq. (1) into two orthogonally related relativistic Euler-type equations, i.e., $u_\mu\partial_\nu T^{\mu\nu}=0$ and $\Delta_{\alpha\mu}\partial_\nu T^{\mu\nu}=0$.

Applying Eq. (31) to the density conservation (2), the projected equations of the energy–momentum conservation, and the Josephson equation (5) respectively, we can obtain the following equations gathering the discontinuities ω^α, υ^α, ψ, and χ,

$$\frac{\partial^2 P}{\partial T\partial\mu}\chi L+\frac{\partial^2 P}{\partial\mu^2}\psi L+n\omega^\mu N_\mu+f^2\upsilon^\mu N_\mu=0, \tag{33}$$

$$\begin{aligned}
&-\left(\mu\frac{\partial^2 P}{\partial T\partial\mu}+T\frac{\partial^2 P}{\partial T^2}\right)\chi L-\left(T\frac{\partial^2 P}{\partial T\partial\mu}+\mu\frac{\partial^2 P}{\partial\mu^2}\right)\psi L\\
&\quad-(\epsilon+P)\omega^\mu N_\mu+u_\mu f^2\upsilon^\mu N_\nu\xi^\nu+u_\mu f^2\xi^\mu\upsilon^\mu N_\mu=0,
\end{aligned} \tag{34}$$

$$\begin{aligned}
&\frac{\partial P}{\partial T}\chi N_\alpha+\frac{\partial P}{\partial T}\chi u_\alpha L+\frac{\partial P}{\partial\mu}\psi N_\alpha+\frac{\partial P}{\partial\mu}\psi u_\alpha L+(\epsilon+P)\omega_\alpha L\\
&\quad+f^2\upsilon_\alpha N_\nu\xi^\nu+f^2\xi_\alpha\upsilon^\mu N_\mu+f^2\xi^\nu\upsilon^\mu N_\mu u_\alpha u_\nu\\
&\quad+f^2\upsilon^\mu N_\nu\xi^\nu u_\alpha u_\mu=0,
\end{aligned} \tag{35}$$

$$\ell^2\psi+L\upsilon^\mu N_\mu+L^3\upsilon^\mu N_\mu=0, \tag{36}$$

where $L\equiv N_\mu u^\mu$ and $\ell^2\equiv 1+L^2$. In Eq. (36), we have also used the projector $\Delta^{\mu\nu}$ to act on the total derivative of Eq. (5). Multiplying N^α to Eq. (35), taking into account $L'\equiv N_\mu u^\mu_s=L$ and $u_\mu\upsilon^\mu=LN_\mu\upsilon^\mu$ for υ^μ, we discard L^3-terms and keep L-terms only in the low speed approximation as a decoupling condition, then obtain the following homogeneous system of equations in the matrix form

$$\begin{pmatrix}
n & f^2 & \frac{\partial^2 P}{\partial\mu^2}L & \frac{\partial^2 P}{\partial T\partial\mu}L\\
-(\epsilon+P) & -\mu f^2 & BL & AL\\
(\epsilon+P)L & \mu f^2 L & \frac{\partial P}{\partial\mu}\ell^2 & \frac{\partial P}{\partial T}\ell^2\\
0 & L & \ell^2 & 0
\end{pmatrix}
\begin{pmatrix}
\omega_N\\ \upsilon_N\\ \psi\\ \chi
\end{pmatrix}=0, \tag{37}$$

where $A\equiv-(\mu\frac{\partial^2 P}{\partial T\partial\mu}+T\frac{\partial^2 P}{\partial T^2})$, $B\equiv-(T\frac{\partial^2 P}{\partial T\partial\mu}+\mu\frac{\partial^2 P}{\partial\mu^2})$, $\omega_N\equiv\omega^\mu N_\mu$ and $\upsilon_N\equiv\upsilon^\mu N_\mu$, the thermodynamic relations $\frac{\partial P}{\partial\mu}=n$ and $\frac{\partial P}{\partial T}=s$ can also be used. To get the non-zero solutions for the discontinuities, the determinant of the coefficient matrix has to be zero and then the following equation is found

$$\alpha L^4+\beta\ell^2L^2+\gamma\ell^4=0, \tag{38}$$

where

$$\alpha = Tw\left[\left(\frac{\partial^2 P}{\partial T^2}\right)\left(\frac{\partial^2 P}{\partial \mu^2}\right) - \left(\frac{\partial^2 P}{\partial T\partial \mu}\right)^2\right], \tag{39}$$

$$\beta = -\left(\frac{\partial^2 P}{\partial T^2}\right)T\left(n^2 + Tsf^2\right) - \left(\frac{\partial^2 P}{\partial \mu^2}\right)s^2 T + \frac{\partial^2 P}{\partial T\partial \mu}\left(2sTn - \mu sTf^2\right), \tag{40}$$

$$\gamma = f^2 s^2 T, \tag{41}$$

and $w = \epsilon + P$. These forms are in agreement with those of Ref. [30]. It is well-known that the local speed of propagation of the shock surface, called wave velocity, measured at a point of Σ where u^μ is the tangent vector to the world-line of the observer, can expressed by [52,73,74]

$$v = \frac{L}{\sqrt{1+L^2}} = \frac{L}{\ell}. \tag{42}$$

In the following, we would denote the wave velocity to be v, having no meaning of that in Section 2. The velocity v of propagating waves (42) is obtained based on the surface theory [52,74]. It is a geometrical result independent of the component substance of the medium. So such a formula can be applied to both normal fluids and superfluids. This velocity form is indeed effective for the calculation of the so-called longitudinal modes, like the first sound and the second sound, including both mechanic and thermodynamic waves. But it cannot be used directly to study the transverse one, or attenuation, in the system with dissipative effects in the first order theory. If we were forced to use it to the transverse case, no attenuation but some unphysical longitudinal mode may be approached. When we analyze the attenuation of the sound, other methods should be taken into account.

Then we can solve Eq. (38) and get two decoupled, well-behaved branches of the propagating solutions. To approach the meaningful result for v^2, both roots should be positive as the requirement of the so-called hyperbolic condition [41]

$$\beta^2 > 4\alpha\gamma > 0 > \alpha\beta, \tag{43}$$

and both roots should be smaller than the speed of light as the requirement of the non-superluminal condition

$$\alpha + \beta + \gamma > 0 > \beta^2 - 4\alpha^2. \tag{44}$$

To identify the significance of the speeds, we use the scaling-invariant equation of state to the system, $P = T^4 g(\mu/T)$ [30,31,33,34], then the solutions of Eq. (38) are

$$v_1^2 = \frac{1}{3},$$
$$v_2^2 = f^2\left[\left(1+\frac{\mu n}{sT}\right)\left(-\frac{3n^2}{sT} + \left(1+\frac{\mu n}{sT}\right)\frac{\partial^2 P}{\partial \mu^2}\right)\right]^{-1}. \tag{45}$$

It is clear that v_1 is the first sound speed associated with the normal sound mode in the system and v_2 is the second sound speed associated with the superflow movement of the system. In the so-called large-entropy limit $Ts \gg \mu n$ together with $\frac{\partial^2 P}{\partial \mu^2} \gg \frac{3n^2}{sT}$, the second sound becomes

$$v_2^2 = f^2\left(\frac{\partial^2 P}{\partial \mu^2}\right)^{-1}, \tag{46}$$

that is, the second sound approaches the fourth sound in the limit of high temperature [31,33].

The sound speeds of relativistic superfluid have been obtained from various approaches theoretically [22,30,42,77,78] by hydrodynamic models and field models. Based on the hydrodynamic description, the propagation of discontinuities gives the correct sound speeds here for the first sound and the second sound. If we take into account the dissipative effects, such as viscous and heat-conducting transports in the system, the sound waves would be attenuated by the dissipative processes [34] and the dispersion equation should give a third root to reflect the characteristic influence of dissipation [6].

4. The growth equation

For the study of the decay and growth of discontinuities propagating in the medium, we should derive the transport or growth equation governing it. Firstly, we should solve ω^α, υ^α and χ in terms of ψ, because these discontinuities are not fully independent of each other,

$$\upsilon_\alpha = -\frac{1}{L}\psi N_\alpha,$$
$$\chi = -\left[\frac{n - (T\frac{\partial n}{\partial T} + \mu\frac{\partial n}{\partial \mu})\frac{L^2}{\ell^2}}{s - (T\frac{\partial s}{\partial T} + \mu\frac{\partial s}{\partial \mu})\frac{L^2}{\ell^2}}\right]\psi,$$
$$\omega_\alpha = \frac{1}{\epsilon+P}\left\{\left[\frac{n - (T\frac{\partial n}{\partial T} + \mu\frac{\partial n}{\partial \mu})\frac{L^2}{\ell^2}}{s - (T\frac{\partial s}{\partial T} + \mu\frac{\partial s}{\partial \mu})\frac{L^2}{\ell^2}}s - n\right]\frac{N_\alpha + Lu_\alpha}{L}\psi + \frac{\mu f^2}{L}\psi N_\alpha\right\}, \tag{47}$$

if the denominator in χ is not zero. When we take the scaling-invariant equation of state and consider the second sound only (the case of the first sound is similar),

$$\upsilon_\alpha = -\frac{1}{L}\psi N_\alpha,$$
$$\chi = -\frac{n}{s}\psi,$$
$$\omega_\alpha = \frac{1}{\epsilon+P}\frac{\mu f^2}{L}\psi N_\alpha. \tag{48}$$

The discontinuities in the second order derivatives in terms of the jump bracket are

$$\left[\frac{\partial^2 T}{\partial x^\alpha \partial x^\beta}\right]N^\alpha N^\beta = \overline{\chi}, \qquad \left[\frac{\partial^2 \mu}{\partial x^\alpha \partial x^\beta}\right]N^\alpha N^\beta = \overline{\psi},$$
$$\left[\frac{\partial^2 u^\alpha}{\partial x^\beta \partial x^\gamma}\right]N^\beta N^\gamma = \overline{\omega}^\alpha, \qquad \left[\frac{\partial^2 \xi^\alpha}{\partial x^\beta \partial x^\gamma}\right]N^\beta N^\gamma = \overline{\upsilon}^\alpha. \tag{49}$$

Using the compatibility conditions for the second partial derivatives (30), we can deduce the system of four equations as follows

$$n\overline{\omega}_N + f^2\overline{\upsilon}_N + \frac{\partial n}{\partial \mu}\overline{\psi}L + \frac{\partial s}{\partial \mu}\overline{\chi}L = \frac{\mu f^2}{\epsilon+P}\left(\frac{\partial n}{\partial \mu} - \frac{\partial s}{\partial \mu}\frac{n}{s}\right)\frac{1}{L}\psi^2 - 2\left(1 - \frac{n\mu}{\epsilon+P}\right)\frac{f^2\Omega}{L}\psi - \frac{\partial s}{\partial \mu}\ell^2\frac{dx}{dt} - \frac{\partial n}{\partial \mu}\ell^2\frac{d\psi}{dt}, \tag{50}$$

$$(\epsilon+P)\overline{\omega}_N L + \mu f^2\overline{\upsilon}_N L - A\overline{\psi}L^2 - B\overline{\chi}L^2 = \left(\frac{n}{s}\right)^2\left(\frac{\partial s}{\partial T}\right)\psi^2 L^2 - 2\frac{\partial s}{\partial \mu}\frac{n}{s}\psi^2 L^2 + \frac{\partial n}{\partial \mu}\psi^2 L^2 - 2\mu f^2 L\ell^2\frac{d\psi}{dt} - 2f^2\psi^2 + \frac{1}{\epsilon+P}\left(\mu f^2\right)^2\psi^2, \tag{51}$$

$$
\begin{aligned}
&(\epsilon+P)\overline{\omega}_N L+\mu f^2\overline{\upsilon}_N L+n\ell^2\overline{\psi}+s\ell^2\overline{\chi}\\
&\quad=\frac{1}{\epsilon+P}\frac{(\mu f^2)^2}{L^2}\psi^2+f^2\frac{\ell^2}{L^2}\psi^2+\mu f^2 L\ell^2\frac{d\psi}{dt}\\
&\qquad-2\frac{1}{\epsilon+P}\left(\mu f^2\right)^2\psi^2,
\end{aligned}
\tag{52}
$$

$$
\overline{\upsilon}_N L+\overline{\psi}=-\frac{1}{\epsilon+P}\frac{\mu f^2}{L^2}\psi^2+\frac{\ell^2}{L}\frac{d\psi}{dt},
\tag{53}
$$

where $\overline{\omega}_N\equiv\overline{\omega}^\mu N_\mu$, $\overline{\upsilon}_N\equiv\overline{\upsilon}^\mu N_\mu$ and Ω is the mean curvature of surface Σ. In the local instantaneous rest frame it can be shown that for a discontinuity Ψ,

$$
\begin{aligned}
&a^{AB}u^\alpha\Psi_{,A}x_{\alpha,B}=\ell^2\frac{d\Psi}{dt},\\
&a^{AB}N^\alpha_{,A}\Psi x_{\alpha,B}=-2\Omega\Psi.
\end{aligned}
\tag{54}
$$

Denoting the quantities on the right hand sides of Eqs. (50), (51), (52) and (53) to be $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ respectively, for the second sound, the following relation should be satisfied

$$
\begin{aligned}
&w\left[\mu L\mathcal{A}-(\mathcal{B}+\mathcal{C})\frac{\mu}{s}\frac{\partial s}{\partial\mu}L^2\right]\\
&\quad-\mu n\left[\mathcal{C}-\mu f^2\mathcal{D}-(\mathcal{B}+\mathcal{C})\ell^2\right]=0.
\end{aligned}
\tag{55}
$$

In the large-entropy limit, we can get a meaningful equation in its standard style [57,72]

$$
c\frac{d\psi}{dt}-\Omega\psi+d\psi^2=0,
\tag{56}
$$

where $c\equiv\frac{\mu n\ell^2}{2wL}$ and $d\equiv\frac{\mu f^2\ell^2}{2wL^2}$. The distance ds traveled by the wave along its normal trajectory in time dt is given by $ds=v_0\,dt$, and v_0 is the constant speed of the second sound, then $s=v_0 t$.

Solving Eq. (56), we can obtain the solution for ψ in the integral form

$$
\psi=\psi_0 J(t)\left[1+\psi_0\left(\frac{d}{c}\right)\int_0^t J(t')\,dt'\right]^{-1},
\tag{57}
$$

where

$$
J(t)=\left(1-2\Omega_0 v_0 t+K_0 v_0^2 t^2\right)^{-\frac{1}{2cv_0}}.
\tag{58}
$$

The mean curvature Ω at any point of the wave surface Σ in terms of s is given by [79,55]

$$
\Omega=\frac{\Omega_0-K_0 s}{1-2\Omega_0 s+K_0 s^2}
\tag{59}
$$

where $\Omega_0=\frac{K_1+K_2}{2}$ and $K_0=K_1K_2$, are the initial values at t_0 of the mean and Gaussian curvatures respectively, and K_1 and K_2 are principle curvatures.

To discuss the solution (57), we consider the special cases for diverging waves only, those on converging waves are similar.

The diverging waves are those K_1 and K_2 are neither positive. Then it is clear from Eqs. (57) and (58) that for an expansive wave front ($\psi_0>0$), $\psi\to 0$ as $t\to\infty$. That is, the expansion waves will decay in time and be damped out finally.

On the other hand, for a compressive wave front ($\psi_0<0$), there would exist a positive critical value ψ_c given by

$$
\psi_c=\left[\frac{d}{c}\lim_{t\to\infty}\int_0^t\frac{dt'}{(1-2\Omega_0 v_0 t'+K_0 v_0^2 t'^2)^{\frac{1}{2cv_0}}}\right]^{-1}.
\tag{60}
$$

If the initial discontinuity $|\psi_0|<\psi_c$, $\psi\to 0$ as $t\to\infty$, the waves will also decay in time and be damped out to zero ultimately. But for the waves with $|\psi_0|>\psi_c$, there exists a critical time t_c satisfying

$$
\int_0^{t_c}\frac{dt'}{(1-2\Omega_0 v_0 t'+K_0 v_0^2 t'^2)^{\frac{1}{2cv_0}}}=\left(\frac{d}{c}|\psi_0|\right)^{-1},
\tag{61}
$$

such that $\psi\to\infty$ as $t\to t_c$, and the waves will grow to infinity in a finite time. In the following, we would discuss three specialized cases for the plane wave ($K_1=K_2=0$), the cylindrical wave ($K_1=-\frac{1}{R_0}$, $K_2=0$), and the spherical wave ($K_1=K_2=-\frac{1}{R_0}$) respectively to interpret Eqs. (60) and (61) more concretely.

For the plane wave, $\Omega_0=K_0=0$. If $\psi_0>0$, $\psi\to 0$ as $t\to\infty$, it is just the case of decay. But if $\psi_0<0$, $t_c=(\frac{d}{c}|\psi_0|)^{-1}$, $\psi\to\infty$ as $t\to t_c$, it is the case of growth.

For the cylindrical wave, $\Omega_0=-\frac{1}{2R_0}$, and $K_0=0$. If $2cv_0>1$, then for an expansive wave front ($\psi_0>0$), $\psi\to 0$ as $t\to\infty$, the discontinuity is decaying in time; while for a compressive wave front ($\psi_0<0$), there exists a finite time t_c

$$
t_c=\frac{R_0}{v_0}\left\{\left[|\psi_0|^{-1}\left(\frac{c}{d}\right)\left(\frac{v_0}{R_0}\right)\left(\frac{2cv_0-1}{2cv_0}\right)+1\right]^{\frac{2cv_0}{2cv_0-1}}-1\right\},
\tag{62}
$$

such that $\psi\to\infty$ as $t\to t_c$, and the discontinuity grows infinitely in a finite time. If $2cv_0<1$, then for $\psi_0>0$, $\psi\to 0$ as $t\to\infty$, the decaying case; while for $\psi_0<0$, there exists a positive ψ_c

$$
\psi_c=\left[\left(\frac{d}{c}\right)\left(\frac{R_0}{v_0}\right)\left(\frac{2cv_0}{1-2cv_0}\right)\right]^{-1},
\tag{63}
$$

such that $\psi\to 0$ as $t\to\infty$ when $|\psi_0|<\psi_c$.

For the spherical wave, $\Omega_0=-\frac{1}{R_0}$, and $K_0=\frac{1}{R_0^2}$, the discussion is similar to that of the cylindrical wave, and the discontinuity ψ is given by

$$
\psi=\psi_0\frac{(1+\frac{v_0}{R_0}t)^{-\frac{1}{cv_0}}}{1+\psi_0(\frac{d}{c})(\frac{R_0}{v_0})(\frac{cv_0}{cv_0-1})[(1+\frac{v_0 t}{R_0})^{\frac{cv_0-1}{cv_0}}-1]}.
\tag{64}
$$

5. Summary

Based on the hydrodynamics for relativistic superfluid, we have studied the evolution of shock wave and the propagation of discontinuity. The general features of shock waves are presented and two sound modes for relativistic superfluid are identified. Moreover, the growth equation is also obtained in an integral form and is used to discuss the growth and decay behaviors of the discontinuities in plane wave, cylindrical wave and spherical wave.

Further studies could be carried out in some directions. One of them is the multi-constituent fluid with more than two constituents as those in superfluid, including superconductive–superfluid mixture [36] and ^{3}He–^{4}He solution [6]. Anisotropic effects of shock waves and the behaviors of high-order discontinuities for relativistic hydrodynamics can also be studied.

Acknowledgements

The author is grateful to Prof. Fan Wang, Dr. Wei Chen, Dr. Yu Jiang, Dr. Feng-Yao Hou, Dr. Xiang He, Dr. Hong-Tao Feng, Dr. Deng-Ke He, Dr. Fei Hu, Dr. Min He, Dr. Yan Yan, Dr. Lei Feng, Dr. Yu-Peng Yang, Dr. Peng Dong and Mr. Jian Wang for useful discussion, and also to Mr. Yi-Qiao Dong, Prof. Tan Lu, and N. Hanzawa. This work is supported in part by the National Natural Science Foundation of China (Grant Nos. 11373068, 10973039 and 10447114), by

Jiangsu Planned Projects for Postdoctoral Research Funds, and by China Postdoctoral Science Foundation Funded Project (Grant No. 2003033010).

References

[1] L.D. Landau, J. Phys. USSR 5 (1941) 71, Zh. Eksp. Teor. Fiz. 11 (1941) 592.
[2] L.D. Landau, E.M. Lifshitz, Fluid Mechanics, 2nd edition, Pergamon, Oxford, 1987.
[3] L. Tisza, Nature 141 (1938) 913.
[4] F. London, Superfluids, vol. 1, Wiley, New York, 1950;
F. London, Superfluids, vol. 2, Wiley, New York, 1954.
[5] S.J. Putterman, Superfluid Hydrodynamics, North-Holland, Amsterdam, 1974.
[6] B.N. Esel'son, M.I. Kaganov, É.Ya. Rudavskiĭ, I.A. Serbin, Sov. Phys. Usp. 17 (1974) 215.
[7] I.M. Khalatnikov, An Introduction to the Theory of Superfluidity, Addison–Wesley, Redwood City, 1989.
[8] W. Israel, Ann. Phys. (N. Y.) 100 (1976) 310;
W. Israel, Phys. Lett. A 86 (1981) 79.
[9] W.G. Dixon, Thermodynamics of Superfluids via Relativity, Churchill College, Cambridge, 1981;
W.G. Dixon, Arch. Ration. Mech. Anal. 80 (1982) 159.
[10] G.V. Vlasov, Sov. Phys. JETP 84 (1997) 729, Zh. Eksp. Teor. Fiz. 111 (1997) 1320;
G.V. Vlasov, arXiv:hep-ph/9808250;
G.V. Vlasov, arXiv:hep-ph/0008219.
[11] S. Shapiro, S. Teukolsky, Black Holes, White Dwarfs and Neutron Stars, Wiley, New York, 1983.
[12] M.E. Gusakov, N. Andersson, Mon. Not. R. Astron. Soc. 372 (2006) 1776, arXiv:astro-ph/0602282.
[13] M.E. Gusakov, E.M. Kantor, P. Haensel, Phys. Rev. C 79 (2009) 055806, arXiv:0904.3467 [astro-ph].
[14] N. Andersson, C. Krueger, G.L. Comer, L. Samuelsson, arXiv:1212.3987 [gr-qc].
[15] B. Carter, in: D. Blaschke, N.K. Glendenning, A. Sedrakian (Eds.), Physics of Neutron Star Interiors, Springer, Berlin, 2001, arXiv:astro-ph/0101257.
[16] D.M. Brink, R.A. Broglia, Nuclear Superfluidity, Cambridge University Press, Cambridge, 2005.
[17] M. Alford, K. Rajagopal, F. Wilczek, Nucl. Phys. B 537 (1999) 443, arXiv:hep-ph/9804403.
[18] I.M. Khalatnikov, V.V. Lebedev, Phys. Lett. A 91 (1982) 70.
[19] E. Trojan, G.V. Vlasov, arXiv:1108.0956 [nucl-th];
G.V. Vlasov, Phys. Lett. A 231 (1997) 149.
[20] V.V. Lebedev, I.M. Khalatnikov, Sov. Phys. JETP 56 (1982) 923, Zh. Eksp. Teor. Fiz. 83 (1982) 1601.
[21] B. Carter, I.M. Khalatnikov, Phys. Rev. D 45 (1992) 4536;
B. Carter, I.M. Khalatnikov, Ann. Phys. (N. Y.) 219 (1992) 243.
[22] M.G. Alford, S.K. Mallavarapu, A. Schmitt, S. Stetina, Phys. Rev. D 87 (2013) 065001, arXiv:1212.0670 [hep-ph].
[23] J.H. Scholtz, E.O. McLean, I. Rudnick, Phys. Rev. Lett. 32 (1974) 147.
[24] D.T. Son, Int. J. Mod. Phys. A 16S1C (2001) 1284, arXiv:hep-ph/0011246.
[25] I.E. Dzyaloshinskii, G.E. Volovick, Ann. Phys. (N. Y.) 125 (1980) 67;
D.T. Son, Phys. Rev. Lett. 84 (2000) 3771, arXiv:hep-ph/9912267.
[26] C. Duval, Z. Horvath, P.A. Horvathy, L. Martina, P.C. Stichel, Mod. Phys. Lett. B 20 (2006) 373, arXiv:cond-mat/0506051;
D.T. Son, N. Yamamoto, Phys. Rev. Lett. 109 (2012) 181602, arXiv:1203.2697 [cond-mat].
[27] C. Pujol, D. Davesne, Phys. Rev. C 67 (2003) 014901, arXiv:hep-ph/0204355.
[28] S. Zhang, Phys. Lett. A 307 (2003) 93, arXiv:hep-ph/0206234.
[29] S. Lin, Nucl. Phys. A 873 (2012) 28, arXiv:1104.5245 [hep-ph];
S. Lin, Phys. Rev. D 85 (2012) 045015, arXiv:1112.3215 [hep-ph].
[30] C.P. Herzog, P.K. Kovtun, D.T. Son, Phys. Rev. D 79 (2009) 066002, arXiv:0809.4870 [hep-th].
[31] A. Yarom, J. High Energy Phys. 0907 (2009) 070, arXiv:0903.1353 [hep-th].
[32] C.P. Herzog, J. Phys. A 42 (2009) 343001, arXiv:0904.1975 [hep-th].
[33] C.P. Herzog, A. Yarom, Phys. Rev. D 80 (2009) 106002, arXiv:0906.4810 [hep-th].
[34] C.P. Herzog, N. Lisker, P. Surowka, A. Yarom, J. High Energy Phys. 1108 (2011) 052, arXiv:1101.3330 [hep-th].
[35] M.G. Alford, S.K. Mallavarapu, A. Schmitt, S. Stetina, arXiv:1212.4410 [hep-ph].
[36] B. Carter, D. Langlois, Nucl. Phys. B 531 (1998) 478, arXiv:gr-qc/9806024.
[37] N. Andersson, G.L. Comer, Living Rev. Relativ. 10 (2007) 1, arXiv:gr-qc/0605010.
[38] Ya.B. Zel'dovich, Yu.P. Raizer, in: W.D. Hayes, R.F. Probstein (Eds.), Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, vol. 1, Academic Press, New York, 1966;
Ya.B. Zel'dovich, Yu.P. Raizer, in: W.D. Hayes, R.F. Probstein (Eds.), Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena, vol. 2, Academic Press, New York, 1967.
[39] L.P. Csernai, Introduction to Relativistic Heavy Ion Collisions, Jonh Wiley & Sons, Chichester, 1994;
L.P. Csernai, M. Gong, Phys. Rev. D 37 (1988) 3231.
[40] P. Danielewicz, P.V. Ruuskanen, Phys. Rev. D 35 (1987) 344.
[41] B. Carter, in: A.M. Anile, Y. Choquet-Bruhat (Eds.), Relativistic Fluid Dynamics, Noto, 1987, Springer-Verlag, Berlin, 1989.
[42] B. Carter, D. Langlois, Phys. Rev. D 51 (1995) 5855.
[43] E. Trojan, G.V. Vlasov, Phys. Rev. D 85 (2012) 107303, arXiv:1102.5659 [hep-th];
E. Trojan, G.V. Vlasov, Zh. Eksp. Teor. Fiz. 140 (2011) 66, arXiv:1103.0673 [hep-th].
[44] H. Stöcker, W. Greiner, Phys. Rep. 137 (1986) 277.
[45] K.A. Bugaev, M.I. Gorenstein, W. Greiner, Heavy Ion Phys. 10 (1999) 333, arXiv:nucl-th/0009035.
[46] M. Cissoko, Phys. Rev. D 55 (1997) 4555.
[47] M.I. Gorenstein, H.G. Miller, R.M. Quick, R.A. Ritchie, Phys. Lett. B 340 (1994) 109, arXiv:hep-ph/9406282.
[48] M.N. Chernodub, H. Verschelde, V.I. Zakharov, Theor. Math. Phys. 170 (2012) 211;
H. Verschelde, V.I. Zakharov, arXiv:1106.4154 [hep-th].
[49] T. Kalaydzhyan, Nucl. Phys. A 913 (2013) 243, arXiv:1208.0012 [hep-ph].
[50] R.P. Feynman, Statistical Mechanics, Benjamin–Cummings, Reading, MA, 1972.
[51] T.Y. Thomas, J. Math. Anal. Appl. 7 (1963) 225.
[52] A.H. Taub, Annu. Rev. Fluid Mech. 10 (1978) 301.
[53] N. Schopohl, D. Waxman, J. Phys. Condens. Matter 4 (1992) L639.
[54] T.Y. Thomas, J. Math. Mech. 6 (1957) 311.
[55] T.Y. Thomas, Concepts from Tensor Analysis and Differential Geometry, Academic Press, New York, 1965.
[56] C. Truesdell, R. Toupin, The classical field theories, in: S. Flügge (Ed.), Encyclopedia of Physics, vol. III/1, Springer-Verlag, Berlin, 1960.
[57] T.Y. Thomas, J. Math. Mech. 6 (1957) 455.
[58] C.N. Kaul, J. Math. Mech. 10 (1961) 393;
C.N. Kaul, J. Math. Anal. Appl. 11 (1965) 482.
[59] T.Y. Thomas, J. Math. Mech. 7 (1958) 893.
[60] V.D. Sharma, Acta Mech. 38 (1981) 249.
[61] R. Shankar, M. Prasad, Arch. Ration. Mech. Anal. 75 (1981) 169;
R. Shankar, M. Prasad, Appl. Sci. Res. 35 (1979) 151.
[62] S. Srinivasan, Indian J. Pure Appl. Math. 12 (1981) 544.
[63] A. Palumbo, Acta Mech. 40 (1981) 141.
[64] A. Rai, M. Gaur, Astrophys. Space Sci. 108 (1985) 175.
[65] T. Nagy, Acta Mech. 101 (1993) 175.
[66] A.R. Elcrat, Int. J. Eng. Sci. 15 (1977) 29.
[67] G.A. Nariboli, J. Math. Phys. Sci. 2 (1968) 294;
G.A. Nariboli, J. Math. Anal. Appl. 16 (1966) 108;
G.A. Nariboli, S.N. Singh, M.P. RangaRao, Proc. Indian Acad. Sci. A 68 (1968) 149.
[68] A.V. Gopalakrshna, M.S. Ganagi, Astrophys. Space Sci. 120 (1986) 139;
A.V. Gopalakrshna, Ann. Inst. Henri Poincaré A 27 (1977) 367;
A.V. Gopalakrshna, Ann. Inst. Henri Poincaré A 38 (1983) 401.
[69] R. Flaccomio, V. La Rosa, Europhys. Lett. 89 (2010) 54001;
R. Giambò, V. La Rosa, G. Muscianisi, Rep. Math. Phys. 67 (2011) 375.
[70] G.A. Maugin, Commun. Math. Phys. 53 (1977) 233.
[71] M.P. Galipò, Nuovo Cimento 38 (1983) 427.
[72] R. Ram, H.N. Singh, M. Gaur, Phys. Scr. 22 (1980) 304;
R. Ram, H.N. Singh, Int. J. Theor. Phys. 19 (1980) 563;
H.N. Singh, R. Ram, J. Phys. A 15 (1982) 1029.
[73] A. Lichnerowicz, Relativistic Hydrodynamics and Magnetohydrodynamics, Benjamin, New York, 1967;
A. Lichnerowicz, J. Math. Phys. 17 (1976) 2135.
[74] G.A. Maugin, Ann. Inst. Henri Poincaré A 24 (1976) 213.
[75] T. Hirano, N. van der Kolk, A. Bilandzic, Hydrodynamics and flow, in: The Physics of the Quark–Gluon Plasma, in: Lect. Notes Phys., vol. 785, Springer-Verlag, Berlin, 2010.
[76] E. Gourgoulhon, in: M. Rieutord, B. Dubrulle (Eds.), Stellar Fluid Dynamics and Numerical Simulations, EDP Sciences, Les Ulis, 2006.
[77] U. Ben-Ya'acov, Phys. Rev. D 52 (1995) 3688.
[78] S.I. Vil'chinskiĭ, Sov. Phys. JETP 79 (1994) 772, Zh. Eksp. Teor. Fiz. 106 (1994) 1430.
[79] E.P. Lane, Metric Differential Geometry of Curves and Surfaces, University of Chicago Press, Chicago, 1940.